



Inhomogeneous spacetimes in Weyl integrable geometry with matter source

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Abstract We investigate the existence of inhomogeneous exact solutions in Weyl Integrable theory with a matter source. In particular, we consider the existence of a dust fluid source while for the underlying geometry we assume a line element which belongs to the family of silent universes. We solve explicitly the field equations and we find the Szekeres spacetimes in Weyl Integrable theory. We show that only the isotropic family can describe inhomogeneous solutions where the LTB spacetimes are included. A detailed analysis of the dynamics of the field equations is given where the past and future attractors are determined. It is interesting that the Kasner spacetimes can be seen as past attractors for the gravitation models, while the unique future attractor describes the Milne universe similar with the behaviour of the gravitational model in the case of General Relativity.

1 Introduction

Analytical and exact solutions play a significant role in the study of gravitational physics. The existence of exact spacetimes is essential in order to understand the physical properties and the nature of the physical space. Inhomogeneous and anisotropic exact spacetimes that have zero magnetic Weyl tensor are very useful in gravitation and cosmology. They include an important family of spacetimes known as the Szekeres universes. The Szekeres spacetimes are the most general cosmological exact solutions of general relativity with a pressureless fluid source [1,2]. They possess no symmetries but the spatial three-slices have a special geometrical structure. In the Szekeres spacetimes, information does not propagate

via gravitational or sound waves, so, they are also known as 'silent' universes [5].

Szekeres spacetimes are inhomogeneous universes that do not admit any vector field isometry. Moreover, the rotation and acceleration of the fluid source must be identically zero, and the pressure constant. In practice, this means the only inhomogeneous matter sources allowed are dust, with or without a cosmological constant. While, in general, the spacetimes are anisotropic – which means that the shear is non-zero and the expansion rate is non-zero. The inhomogeneous Szekeres spacetimes are classified into two families: the inhomogeneous Kantowski–Sachs (-like) spacetimes and the inhomogeneous FLRW (-like) spacetimes.

There are applications of Szekeres spacetimes in gravitational physics and cosmology [1,6]. A complete description of the scalar polynomial curvature singularities in both classes of Szekeres solution have been established, and they are velocity-dominated. They have Newtonian counterparts and contain no gravitational waves [4,10]. In addition, the asymptotic behavior in the distant future has been analyzed [3,7].

A more general gravitational collapse, known as quasi-spherical by using the Szekeres spacetimes was studied in Ref. [8], where it was found that a strong radial increase in the density, the fluid heralds the onset of a naked singularity. The matter distribution in Szekeres spaces has a dipolar character [9], while there is no gravitational radiation emission from the inhomogeneous moving dust [10], for other applications of Szekeres spacetimes in gravitational physics we refer the reader to [11–14] and references therein. Tilted Szekeres models were studied in Ref. [15] where it was found that vorticity follows the congruence of the fluid world lines. Recently, the frame rotation of the Szekeres spacetimes which relates the cosmological solutions with the quasi-spherical exact solutions was studied in Ref. [16].

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The quasi-spherical Szekeres dust solutions are a generalization of the spherically symmetric Lemaître–Tolman–Bondi dust models where the spherical shells of constant mass are not concentric. A coordinate-independent analysis of the dynamics of the spherically symmetric Lemaître–Tolman–Bondi cosmologies, emphasizing their relation to the Friedmann Lemaître cosmologies was given in [17]. In general, it was shown that ever-expanding Lemaître–Tolman–Bondi cosmologies isotropize at late times, approaching the de Sitter universe, or the Milne universe, depending on whether or not a cosmological constant is present. For the analysis, a dimensionless scalar is introduced to represent the ratio of the Weyl and Ricci curvatures. In all cases, there is a finite limit at late times, its value determines the asymptotic spatial inhomogeneities in various physical quantities. The Lemaître–Tolman–Bondi cosmologies for which the initial singularity is isotropic were also identified. The collapsing quasi-spherical Szekeres dust solution, where an apparent horizon covers all shell-crossings that will occur, can be considered as a model for the formation of a black hole. The apparent horizon can be detected by a Cartan invariant [18]. In the former reference, solutions of this sort are reviewed together with their spin coefficients and curvature scalars in the Newman–Penrose formalism. The Cartan–Karlhede algorithm is used to generate the minimal set of extended Cartan invariants. Cartan scalars are compared with the kinematic scalars [19] and q-scalars [20], which are two well-known sets of scalars used to characterize Szekeres solutions.

Inhomogeneous spacetimes can be seen as limits of FLRW spacetimes with inhomogeneous perturbations, such a comparison between non-spherical Szekeres spaces and the dynamics of cosmological perturbation theory was performed in Ref. [21]. Specifically, it was proved that the linearised Szekeres evolution equations and their solutions fully coincide with the corresponding equations of the linear cosmological perturbation theory and their solutions in the isochronous comoving gauge. Moreover, the conservation of the curvature perturbation holds for the appropriate linear approximation of the exact Szekeres fluctuations in Λ -cosmology, while the different collapse morphologies of Szekeres models yield different growth factors to those that follow from the analysis of redshift space distortions.

There are various generalizations of the Szekeres exact solutions where additional matter sources contribute to the gravitating matter [2]. Indeed, the first generalization presented by Szafron in Ref. [22], where the dust fluid source was replaced by a perfect fluid with non-zero pressure, leading to the Szekeres–Szafron spacetimes. The cosmological constant term was introduced by Barrow et al. [23] where the inhomogeneous analogue of the Λ CDM model was derived. Other kinds of matter source have been introduced, such as heat flow, electromagnetic field, viscosity, and an aether field in the context of Einstein-aether theory [24–31].

In this work, we are interested in determining the exact inhomogeneous spacetimes in Weyl Integrable theory [32]. A Weyl manifold is a conformal manifold equipped with a connection which preserves the conformal structure and is torsion-free. In Weyl Integrable theory the connection structure is related to the Levi-Civita connection, to which it differs by a scalar field of the conformal metric. Specifically, if $g_{\mu\nu}$ is a metric tensor with Levi-Civita connection $\Gamma_{\mu\nu}^\kappa$, then in Weyl Integrable theory the manifold is supported by the set $\{g_{\mu\nu}, \tilde{\Gamma}_{\mu\nu}^\kappa\}$ where $\tilde{\Gamma}_{\mu\nu}^\kappa$ is the Levi-Civita connection for the conformally related metric $\tilde{g}_{\mu\nu} = \phi g_{\mu\nu}$, where ϕ is a scalar field. An important characteristic of the Weyl Integrable theory is that it is in agreement with current astronomical and other observations [33].

Physical consequences of Weyl invariant theories are discussed, e.g., in Refs. [34–36]. In Ref. [37], it is discussed whether or not a general Weyl structure is a suitable mathematical model of spacetime. In this regard, it was found that a Weyl integrable spacetime is the most general structure suitable to model spacetime. The well-posedness of the Cauchy problem for particular kinds of geometric scalar-tensor theories of gravity, which are based on a Weyl integrable spacetime, is given in Ref. [38]. In Ref. [39], a formulation of general relativity on a Weyl-integrable geometry which contains cosmological solutions, exhibiting acceleration in the present cosmic expansion, is studied. The conditions for accelerated expansion of the universe are derived there. A particular solution for the Weyl scalar field describing a cosmological model for the present time is obtained in concordance with the data-combination Planck + WP + BAO + SN. In Ref. [40], the evolution of 4-, 5- and 6-dimensional cosmological models based on the integrable Weyl geometry are considered numerically both for empty spacetime and for scalar field with non-minimal coupling with gravity. In Ref. [41], the motion of massless particles on the background of a toroidal topological black hole is analyzed in the context of conformal Weyl gravity. Null geodesics are found analytically in terms of the Jacobi elliptic functions.

There are various exact solutions of the field equations in Weyl Integrable theory. Vacuum cosmological models were studied in Ref. [44], while higher- or lower-dimensional gravitational models were studied in Refs. [40, 42, 43]. In Ref. [45], the authors studied gravitational models in Weyl Integrable theory with matter source, an electromagnetic field, and additional scalar field. In these models an interaction between the scalar field of the Weyl theory and the matter sources is introduced, when the field equations are written in the covariant form using the tensor quantities of general relativity. Inhomogeneous models in Weyl Integrable models were also studied in Refs. [46–49], while spherically symmetric solutions can be found in Ref. [50]. In addition in Ref.

[50] the authors discuss the similarities and the differences of the Weyl Integrable theory with the Brans–Dicke theory.

The plan of this paper is as follows. In Sect. 2 we present the basic properties and definitions of the gravitational field equations in Weyl Integrable theory. We rewrite the field equations in a way that is equivalent in form to general relativity and show that a scalar field is introduced in the field equations, and we discuss the case where an ideal gas contributes in the gravitational model. For the underlying geometry, we consider the line element which belong to the silent universe class, and describes the Szekeres spacetimes in general relativity. Exact solutions of the field equations are presented in Sect. 4. In Sects. 4 we perform a detailed analysis of the field equations in order to understand the past and future evolution of the cosmological solutions. Finally, in Sect. 5 we discuss our results and draw our conclusions.

2 Weyl integrable gravity

Weyl geometry is an extension of Riemannian geometry, specified by a metric tensor $g_{\mu\nu}$ and a gauge vector field ω_μ . The covariant derivative $\tilde{\nabla}_\mu$ is defined by the (Weyl) affine connection $\tilde{\Gamma}^\kappa_{\mu\nu}$, with the property,

$$\tilde{\nabla}_\kappa g_{\mu\nu} = \omega_\kappa g_{\mu\nu}, \quad (1)$$

from which we infer that the gauge vector ω_μ plays a significant role in the geometry. Specifically, the Weyl affine connection $\tilde{\Gamma}^\kappa_{\mu\nu}$ is related to the Christoffel symbols $\Gamma^\kappa_{\mu\nu}$ of the metric tensor $g_{\mu\nu}$ as follows:

$$\tilde{\Gamma}^\kappa_{\mu\nu} = \Gamma^\kappa_{\mu\nu} - \omega_\mu \delta^\kappa_\nu + \frac{1}{2} \omega^\kappa g_{\mu\nu}. \quad (2)$$

The curvature tensor in the Weyl geometry is defined as

$$\tilde{\nabla}_\nu (\tilde{\nabla}_\mu u_\kappa) - \tilde{\nabla}_\mu (\tilde{\nabla}_\nu u_\kappa) = \tilde{R}_{\kappa\lambda\mu\nu} u^\lambda \quad (3)$$

where, by using (2), we observe that in general $\tilde{R}_{\kappa\lambda\mu\nu}$ is not symmetric as it is in the case of Riemannian geometry.

In this work, we are interested in the case where $\tilde{R}_{\kappa\lambda\mu\nu}$ has the same symmetric properties as in Riemannian geometry. This is true when ω_μ is a gradient vector, which means that there exists a scalar ϕ such that $\omega_\mu = \phi_{,\mu}$. In addition, in this case, length variations are integrable along a closed path. This specific theory is known as Weyl Integrable geometry. Moreover, there exists a conformal map which relates the metric tensor $g_{\mu\nu}$ of a Riemannian space into that of Weyl integrable space, which means that a Weyl integrable space is also conformally a Riemann space.

In Weyl integrable geometry, the Ricci tensor $\tilde{R}_{\mu\nu}$ is related to the Riemannian Ricci tensor $R_{\mu\nu}$ by,

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - \tilde{\nabla}_\nu (\tilde{\nabla}_\mu \phi) - \frac{1}{2} (\tilde{\nabla}_\mu \phi) (\tilde{\nabla}_\nu \phi)$$

$$- \frac{1}{2} g_{\mu\nu} \left(\frac{1}{\sqrt{-g}} (g^{\mu\nu} \sqrt{-g} \phi)_{,\mu\nu} - g^{\mu\nu} (\tilde{\nabla}_\mu \phi) (\tilde{\nabla}_\nu \phi) \right), \quad (4)$$

where the Ricci scalar, in a four-dimensional manifold, is written as [45]

$$\tilde{R} = R - \frac{3}{\sqrt{-g}} (g^{\mu\nu} \sqrt{-g} \phi)_{,\mu\nu} + \frac{3}{2} (\tilde{\nabla}_\mu \phi) (\tilde{\nabla}_\nu \phi). \quad (5)$$

2.1 Gravitational action integral

We define the simple gravitational action Integral, which includes the Weyl Ricci scalar \tilde{R} and the field $\phi_{,\mu}$, as

$$S_W = \int dx^4 \sqrt{-g} \left(\tilde{R} + \xi (\tilde{\nabla}_\nu (\tilde{\nabla}_\mu \phi)) g^{\mu\nu} \right), \quad (6)$$

where ξ is an arbitrary coupling constant. At this point, we remark that

$$\begin{aligned} (\tilde{\nabla}_\nu (\tilde{\nabla}_\mu \phi)) g^{\mu\nu} &= \frac{1}{\sqrt{-g}} (g^{\mu\nu} \sqrt{-g} \phi)_{,\mu\nu} \\ &\quad - 2g^{\mu\nu} (\tilde{\nabla}_\mu \phi) (\tilde{\nabla}_\nu \phi). \end{aligned} \quad (7)$$

Variation with respect to the metric tensor of the action integral, S_W , provides the gravitational field equations [45],

$$\begin{aligned} \tilde{G}_{\mu\nu} + \tilde{\nabla}_\nu (\tilde{\nabla}_\mu \phi) - (2\xi - 1) (\tilde{\nabla}_\mu \phi) (\tilde{\nabla}_\nu \phi) \\ + \xi g_{\mu\nu} g^{\kappa\lambda} (\tilde{\nabla}_\kappa \phi) (\tilde{\nabla}_\lambda \phi) = 0, \end{aligned} \quad (8)$$

where $\tilde{G}_{\mu\nu}$ is the Weyl Einstein tensor. Moreover, variation with respect to the scalar field ϕ gives

$$(\tilde{\nabla}_\nu (\tilde{\nabla}_\mu \phi)) g^{\mu\nu} + 2g^{\mu\nu} (\tilde{\nabla}_\mu \phi) (\tilde{\nabla}_\nu \phi) = 0, \quad (9)$$

that is, a Klein–Gordon equation of the form,

$$g^{\mu\nu} \nabla_\nu \nabla_\mu \phi = 0, \quad (10)$$

where ∇_μ denotes the Riemannian covariant derivative.

The gravitational field equations (8) can be rewritten by using the Riemannian Einstein tensor $G_{\mu\nu}$ as follows

$$G_{\mu\nu} - \lambda \left(\phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi^{,\kappa} \phi_{,\kappa} \right) = 0, \quad (11)$$

in which the new constant λ is defined as $2\lambda \equiv 4\xi - 3$.

Consequently, the field equations (11) for $\lambda > 0$ are those of general relativity¹ with a massless scalar field. A new possibility is introduced when $\lambda < 0$, which correspond to the addition of a massless phantom scalar field.

Until now, we have considered the case of vacuum. Now, we present the field equations in the presence of a matter

¹ We consider the signature of the metric to be $(-, +, +, +)$.

source. Specifically, we consider the cases where a pressureless dust fluid source contributes to the gravitational field equations.

2.2 The presence of dust

When a pressureless fluid dust source is included, the gravitational field equations become

$$G_{\mu\nu} - \lambda \left(\phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi^{,\kappa} \phi_{,\kappa} \right) = T_{\mu\nu}^{(m)}, \quad (12)$$

where $T_{\mu\nu}^{(m)} = e^{-\frac{\phi}{2}} \rho_m u_\mu u_\nu$, while the Klein–Gordon equation (10) becomes

$$\frac{1}{\sqrt{-g}} (g^{\mu\nu} \sqrt{-g} \phi_{,\mu})_{,\nu} - \frac{1}{2\lambda} e^{-\frac{\phi}{2}} \rho_m = 0, \quad (13)$$

while the conservation equation for the matter field reads $\tilde{\nabla}_\nu T^{(m)\mu\nu} = 0$.

We can see that there is a coupling between the scalar field and the dust fluid; hence, the coupling and effective pressure term can depend on the energy density ρ_m . The interaction between scalar field and dust fluid has been proposed as a potential mechanism to explain the cosmic coincidence problem [51–54]. Various interaction models have been studied before in the literature; for instance, see [55–59], and references therein.

3 Inhomogeneous spacetimes

The gravitational model that we have considered in Weyl Integrable geometry is equivalent to that of general relativity with an effective energy-momentum tensor, where the field equations are of the form

$$G_{\mu\nu} = T_{\mu\nu}, \quad (14)$$

and $T_{\mu\nu}$ is the effective energy-momentum tensor. It consists of a massless scalar field ϕ , and an additional fluid source interacting with the field ϕ . In particular, $T_{\mu\nu} = T_{\mu\nu}^{(\phi)} + \hat{T}_{\mu\nu}$, where $\hat{T}_{\mu\nu}$ describes the energy-momentum tensor of a pressureless fluid, i.e. $\hat{T}_{\mu\nu} = T_{\mu\nu}^{(m)}$, or of the second scalar field, ψ ; that is,

$$T_{\mu\nu}^{(\psi)} = e^{-2\phi} \left(\psi_{,\mu} \psi_{,\nu} - \frac{1}{2} g_{\mu\nu} \psi^{,\kappa} \psi_{,\kappa} - g_{\mu\nu} U(\psi) \right), \quad (15)$$

where $T_{\mu\nu}^{(\phi)}$ is the energy-momentum tensor of the massless scalar field,

$$T_{\mu\nu}^{(\phi)} = \lambda \left(\phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi^{,\kappa} \phi_{,\kappa} \right). \quad (16)$$

However, as we discussed before the continuity equation $\nabla_\nu T^{\mu\nu} = 0$, provides $\nabla_\nu (T^{(\phi)\mu\nu} + \hat{T}^{\mu\nu}) = 0$, that is,

$\nabla_\nu (T^{(\phi)\mu\nu}) = Q$, $\nabla_\nu (\hat{T}^{\mu\nu}) = -Q$, where $Q = Q(x^\mu)$, is the interacting term.

In this work, we assume that the underlying spacetime is described by the inhomogeneous and anisotropic diagonal line element

$$ds^2 = -dt^2 + e^{2A} dr^2 + e^{2B} (dy^2 + dz^2), \quad (17)$$

in which $A = A(t, r, y, z)$ and $B = B(t, r, y, z)$. The functional forms of the two scale factors A, B are determined by the solution of the field equations (14).

In the case of general relativity, for $B_{,r} = 0$, i.e. $B = B(t, y, z)$, the exact solutions belong to the Kantowski–Sachs (-like) family, while for $B_{,r} \neq 0$ the resulting spacetimes are inhomogeneous and isotropic. Generalization of the Szekeres spacetimes with a purely time-dependent scalar field have been studied before in Ref. [60]. In particular, a quintessence scalar field was considered and the scalar field should be homogeneous. Although the density can be inhomogeneous in metric (17), the pressure must be homogeneous [2].

Hence, from the results for the vacuum solution of [60] when the scalar field is massless, we recover the exact solution of the vacuum Weyl Integrable geometry for $\lambda > 0$. However, in the presence of an additional matter source, as we have here because of the existence of the interaction term, the analytic solutions will be different.

In the following section, we proceed with the presentation of the analytic solutions for the field equations (14), where the underlying spacetime is described by the line element (17).

We require the pressure term of the effective energy-momentum tensor $T_{\mu\nu}$ in (14) to be homogeneous such that the FLRW limit to be provided. Thus, $\phi = \phi(t)$ while for the matter source we have $\rho_m = \rho_m(t)$. The latter follows easily, if we rewrite the energy-momentum tensor $T_{\mu\nu}$ such that to define a new pressure component in order to eliminate the interaction term. The steps that we follow to solve the field equations are similar to those taken in Ref. [60]. Thus, we omit the presentation and go directly to the main results.

Similarly, in the case of the homogeneous scalar field, we find that the Szekeres-like solutions in the Weyl Integrable theory are classified into two classes of solutions, (A) the inhomogeneous Kantowski–Sachs family of solutions and the (B) inhomogeneous FLRW (-like) solutions.

For the Kantowski–Sachs family of solutions the unknown functions in the line element (17) are $A(t, r, y, z) = \alpha(t)$ and $B(t, r, y, z) = \beta(t) (c_1 uv + c_2 u + c_3 v + c_4)$, where the $y = u + v$, $z = i(u - v)$, so that the line element is written as [60]:

$$ds^2 = -dt^2 + \alpha^2(t) dr^2 + \beta^2(t) e^{2C(y,z)} (dy^2 + dz^2), \quad (18)$$

and the curvature, K , of the two-dimensional surface of constant curvature $\{y - z\}$ to be related with the constants c_1 , c_2 , c_3 and c_4 as follows, $K = c_1 c_4 - c_2 c_3$. The unknown time-dependent functions $\alpha(t)$, $\beta(t)$ are determined by a set of differential equations that will be presented in the following sections.

The second family of solutions which correspond to the inhomogeneous FLRW-like spacetimes are described by the line element [60]:

$$ds^2 = -dt^2 + \alpha^2(t) \left(\left(\frac{\partial C(r, y, z)}{\partial r} \right)^2 dr^2 + e^{2C(r, y, z)} (dy^2 + dz^2) \right). \quad (19)$$

The function $C(r, y, z)$ is now given by $C(y, z) = -2 \ln(\gamma_1(r)uv + \gamma_2(r)u + \gamma_3(r)v + \gamma_4(r))$, where the functions $\gamma_1(r)$, $\gamma_2(r)$, $\gamma_3(r)$, and $\gamma_4(r)$ are constrained by $k = \gamma_1(r)\gamma_4(r) - \gamma_2(r)\gamma_3(r)$, where k is the spatial curvature of the FLRW-like spacetime. The scale factor $\alpha(t)$ is given by the generalized Friedmann equations in Weyl Integrable geometry given below.

3.1 Kantowski–Sachs spacetimes

The unknown scale factors of the Kantowski-Sachs spacetime (18) are given by the following system,

$$\frac{2}{\alpha\beta} \dot{\alpha}\dot{\beta} + \frac{1}{\beta^2} \dot{\beta}^2 + \frac{K}{\beta^2} + \frac{\lambda}{2} \dot{\phi}^2 + e^{-\frac{\phi}{2}} \rho_m = 0, \quad (20)$$

$$\frac{\ddot{\alpha}}{\alpha} + \frac{\ddot{\beta}}{\beta} + \frac{1}{\alpha\beta} \dot{\alpha}\dot{\beta} + \frac{\lambda}{2} \dot{\phi}^2 = 0, \quad (21)$$

$$2 \frac{\ddot{\beta}}{\beta^2} + \frac{\dot{\beta}^2}{\beta^2} - \frac{K}{\beta^2} + \frac{\lambda}{2} \dot{\phi}^2 = 0, \quad (22)$$

while the equation of motion for the scalar field and the matter source are given by,

$$\ddot{\phi} + \left(\frac{\dot{\alpha}}{\alpha} + 2 \frac{\dot{\beta}}{\beta} \right) \dot{\phi} + \frac{1}{2\lambda} e^{-\frac{\phi}{2}} \rho_m = 0, \quad (23)$$

$$\dot{\rho}_m + \left(\frac{\dot{\alpha}}{\alpha} + 2 \frac{\dot{\beta}}{\beta} - \dot{\phi} \right) \rho_m = 0, \quad (24)$$

where overdot means total derivative with respect to the variable t .

3.2 FLRW spacetimes

Analogously, the unique scale factor for the FLRW (-like) spacetime (19) is given by the (modified) Friedmann equations

$$-3 \left(\frac{\dot{\alpha}}{\alpha} \right)^2 + 3k\alpha^{-2} + \frac{\lambda}{2} \dot{\phi}^2 + e^{-\frac{\phi}{2}} \rho_m = 0, \quad (25)$$

$$-2 \frac{\ddot{\alpha}}{\alpha} - \left(\frac{\dot{\alpha}}{\alpha} \right)^2 + k\alpha^{-2} - \frac{\lambda}{2} \dot{\phi}^2 = 0, \quad (26)$$

and the scalar field ϕ satisfies the Klein–Gordon equation,

$$\ddot{\phi} + 3 \frac{\dot{\alpha}}{\alpha} \dot{\phi} + \frac{1}{2\lambda} e^{-\frac{\phi}{2}} \rho_m = 0, \quad (27)$$

while the conservation equation for the dust fluid source is

$$\dot{\rho}_m + \left(3 \frac{\dot{\alpha}}{\alpha} - \dot{\phi} \right) \rho_m = 0. \quad (28)$$

At this point, we remark that for $\lambda = 0$, only the vacuum solutions of general relativity are recovered, while the Szekeres spacetimes are recovered when $\phi = \phi_0$ and $\lambda \rightarrow \infty$. This is reminiscent of the range of the constant Brans–Dicke parameter, ω , in scalar-tensor theory such, where the limit of general relativity to be recovered as $\omega \rightarrow \infty$ [61]. This family of spacetimes includes also the inhomogeneous Lemaître–Tolman–Bondi (LTB) spacetimes [62].

In the following we show the analytic solution for the inhomogeneous FLRW (-like) spacetime.

3.3 Inhomogeneous analytic solution

Now let us consider the case when the spatial curvature is zero, i.e. $k = 0$. The gravitational field equations can be rewritten in an equivalent form,

$$2\dot{H} + 3H^2 + \frac{\lambda}{2} \Phi^2 = 0, \quad (29)$$

$$\dot{\Phi} + 3H\Phi + \frac{3}{2\lambda} H^2 - \frac{1}{4} \Phi^2 = 0, \quad (30)$$

where $H = \frac{\dot{a}}{a}$ is the Hubble function and $\Phi = \dot{\phi}$.

We continue by defining the new variables $\{R, \Theta\}$ which are given by the point transformation

$$H \equiv R \cos \Theta, \quad \Phi \equiv \sqrt{\frac{6}{\lambda}} R \sin \Theta. \quad (31)$$

Therefore, the field equations (29), (30) in the new coordinates are,

$$-4 \sqrt{\frac{\lambda}{6}} \frac{\dot{R}}{R^2} = 3 \sqrt{\frac{6}{\lambda}} \cos \Theta (3 - 2 \cos^2 \Theta) + \sin \Theta (2 \cos^2 \Theta - 1), \quad (32)$$

$$-4 \sqrt{\frac{\lambda}{6}} \frac{\dot{\Theta}}{R} = \cos \Theta (2 \cos^2 \Theta - 1) + \sqrt{6\lambda} \sin \Theta (2 \cos^2 \Theta - 1), \quad (33)$$

from which it follows that the general algebraic solution expressed in parametric form is

$$I_0 = -\frac{(6\lambda - 1)}{2} R^2 + (\sqrt{6\lambda} - 6\lambda) \ln(\sin(\Theta) - \cos(\Theta)) +$$

$$- (\sqrt{6\lambda} + 6\lambda) \ln(\sin(\Theta) + \cos(\Theta)) + (6\lambda + 1) \ln(6\sqrt{\lambda} \sin \Theta + \sqrt{6} \cos \Theta), \quad (34)$$

where I_0 is constant. In the special case where $6\lambda = 1$, the generic algebraic solution follows

$$I_0 = -\frac{1}{2}R^2 - \frac{1}{1 + \tan \Theta} - \ln(\sin^2 \Theta - \cos^2 \Theta). \quad (35)$$

We continue our analysis by studying the dynamics of the field equations, specifically, the ones of the (Weyl) Szekeres system.

The phase space portrait of the field equations (32), (33) is presented in Fig. 1.

4 Dynamical analysis

The field equations (14) with time-derivatives can be written in a covariant form using the kinematic variables for the observer: the volume expansion rate $\theta = 3H$, the shear scalar σ , the electric part of the Weyl tensor E , and the components of the effective fluid energy density ρ and pressure p .

In particular, the field equations are then expressed as follows [19, 63]

$$\dot{\rho} + \theta(\rho + p) = 0, \quad (36a)$$

$$\dot{\theta} + \frac{\theta^2}{3} + 6\sigma^2 + \frac{1}{2}(\rho + 3p) = 0, \quad (36b)$$

$$\dot{\sigma} - \sigma^2 + \frac{2}{3}\theta\sigma + E = 0, \quad (36c)$$

$$\dot{E} + 3E\sigma + \theta E + \frac{1}{2}(\rho + p)\sigma = 0, \quad (36d)$$

with the constraint equation,

$$\frac{\theta^2}{3} - 3\sigma^2 + \frac{{}^{(3)}R}{2} = \rho, \quad (36e)$$

where ${}^{(3)}R$ is the spatial curvature of the three-dimensional hypersurfaces. The latter system is known as the Szekeres–Szafron system and has been widely studied in the literature [5, 64–66].

In Weyl Integrable theory with a dust fluid source the effective energy density and pressure are $\rho = e^{-\frac{\phi}{2}}\rho_m + \rho_\phi$, $p = p_\phi$, in which $\rho_\phi = \frac{\lambda}{2}\dot{\phi}^2$ and $p_\phi = \frac{\lambda}{2}\dot{\phi}^2$. In addition, from Eq. (36a) we can write the equivalent system

$$\dot{\rho}_m + (\theta - \dot{\phi})\rho_m = 0, \quad (37a)$$

$$\dot{\rho}_\phi + \theta(\rho_\phi + p_\phi) + \frac{e^{-\frac{\phi}{2}}}{2}\rho_m\dot{\phi} = 0. \quad (37b)$$

In the following, we rewrite the field equations (36a)–(36e) using expansion-normalized variables to determine the

stationary points of the dynamical system. We remark that every stationary point corresponds to an exact solution of the field equations, which can describe a specific epoch provided by the dynamics of the system. The stability of the stationary points is also determined, which is needed to determine the past and future evolution of the solutions provided by the stationary points.

4.1 Dimensionless variables

We define the new expansion-normalised dimensionless variables

$$\Omega_m = \frac{3e^{-\frac{\phi}{2}}\rho_m}{\theta^2}, \quad \Omega_R = \frac{3R}{2\theta^2}, \quad x = \frac{\sqrt{6}\dot{\phi}}{2\theta}, \quad \beta = \frac{\sigma}{\theta}, \quad \text{and} \quad \alpha = \frac{E}{\theta^2}. \quad (38)$$

In the new variables, the Szekeres system becomes

$$\Omega'_m = \frac{1}{2}\Omega_m(\sqrt{6}x + 8\lambda x^2 + 72\beta^2 + 2(\Omega_m - 1)), \quad (39a)$$

$$x' = \frac{1}{12\lambda}(2\lambda x(36\beta^2 + 4\lambda x^2 + \Omega_m - 4) - \sqrt{6}\Omega_m), \quad (39b)$$

$$\beta' = \frac{1}{2}(6\beta^2(1 + 6\beta) + \beta\Omega - 2(\beta + 3\alpha) + 4\lambda\beta x^2), \quad (39c)$$

$$\alpha' = \frac{1}{2}(2\alpha(\Omega + 4\lambda x^2 - 1 + 9\beta(4\beta - 1)) - \beta(2\lambda x^2 + \Omega_m)), \quad (39d)$$

with (first integral) constraint equation

$$\Omega_R = -1 + 9\beta^2 + \lambda x^2 + \Omega_m, \quad (39e)$$

where the prime derivative is defined by $\Omega'_m \equiv \frac{d\Omega_m}{d\tau}$, where $\tau = \ln a$ and $a(\tau)$ is the geometric mean expansion scale factor ($\dot{a}/a = H$). Moreover, the parameter for the equation of state of an effective fluid source, $w_{tot} = \frac{p}{\rho}$, is expressed in terms of the dimensionless variables as

$$w_{tot} = \frac{1}{3}(\Omega - 1 + 4x^2\lambda + 36\beta^2). \quad (40)$$

4.2 Stationary points

The set of stationary points, \mathbf{P} , have coordinates $\mathbf{P} = (\Omega_m(\mathbf{P}), x(\mathbf{P}), \beta(\mathbf{P}), \alpha(\mathbf{P}))$, and the physical properties of the exact solutions at these points for the four-dimensional dynamical system (39a)–(39b) are presented below.

Point $P_1 = (0, 0, 0, 0)$ describes an empty isotropic universe, with spatial curvature $\Omega_R(P_1) = -1$ and a parameter for the equation of state $w_{tot}(P_1) = -\frac{1}{3}$. From the latter, we infer that the exact solution at the point P_1 is the Milne universe. In order to infer the stability of the exact solution

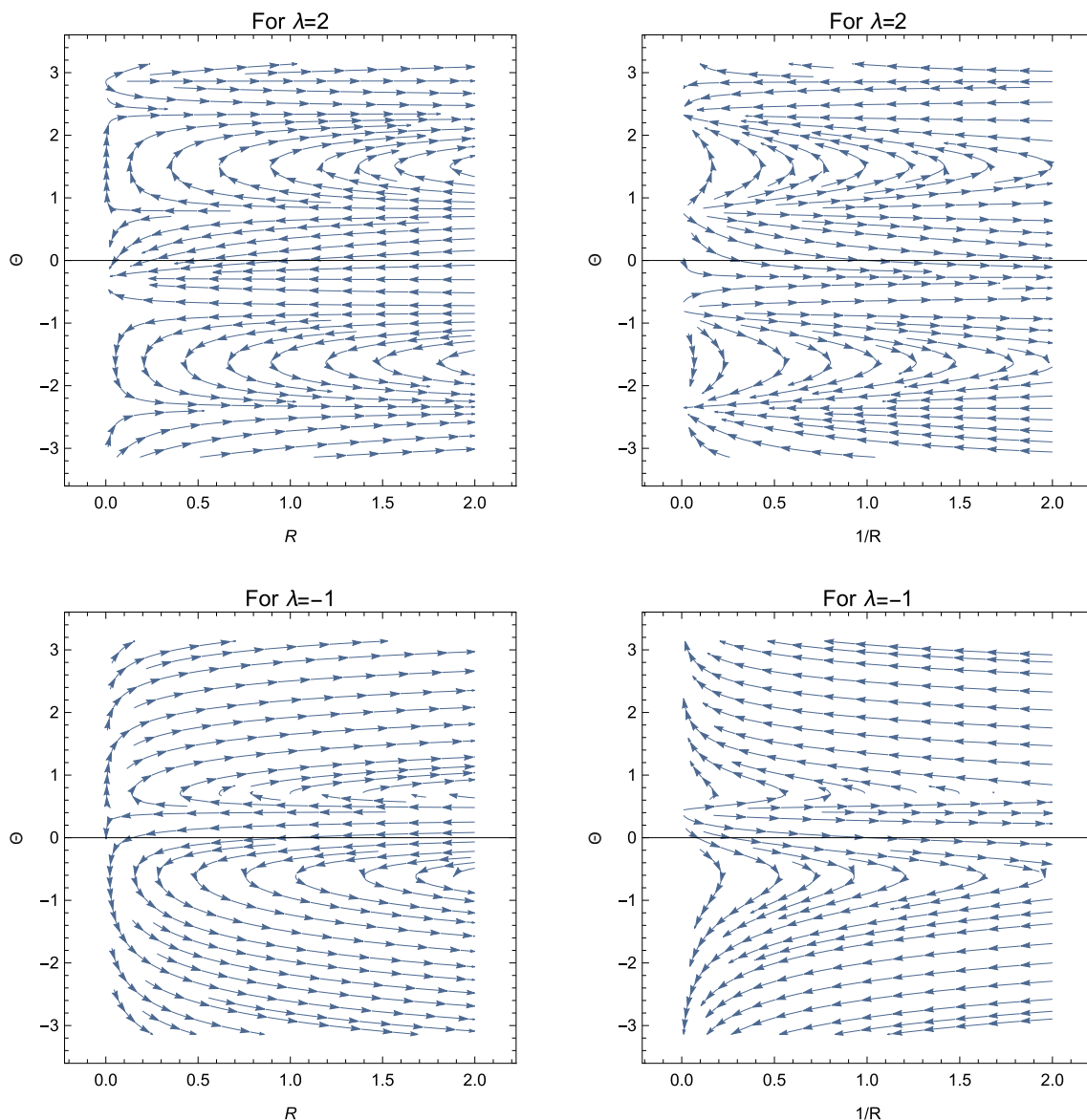


Fig. 1 The phase portrait of the dynamical system (32), (33) for $\lambda = 2$ and $\lambda = -1$. Note that for negative values of λ , we apply the transformation $\Theta \rightarrow i\Theta$

at point P_1 , we determine the eigenvalues of the linearized system around P_1 . They are $e_1(P_1) = -1$, $e_2(P_1) = -1$, $e_3(P_1) = -1$ and $e_4(P_1) = -\frac{2}{3}$, hence P_1 is an attractor.

Point $P_2 = (0, 0, \frac{1}{6}, 0)$ has physical quantities $\Omega_R(P_2) = -\frac{3}{4}$. This point describes an anisotropic Kantowski–Sachs universe. The eigenvalues of the linearized system are $e_1(P_2) = 0$, $e_2(P_2) = -\frac{3}{4}$, $e_3(P_2) = -\frac{1}{2}$ and $e_4(P_2) = \frac{3}{2}$, from which we infer that P_2 is a saddle point, that is, the exact solution at this point is unstable.

Point $P_3 = (0, 0, -\frac{1}{3}, 0)$ describes a vacuum Bianchi I universe, $\Omega_R(P_3) = 0$, and more specifically, the Kasner universe. The eigenvalues of the linearized system are $e_1(P_3) = 0$, $e_2(P_3) = 3$, $e_3(P_3) = 3$ and $e_4(P_3) = 6$, hence P_3 is a source.

Point $P_4 = (0, 0, \frac{1}{3}, \frac{2}{9})$ describes a vacuum Kasner universe, $\Omega_R(P_4) = 0$, and the exact solution is unstable. The eigenvalues are $e_1(P_4) = 0$, $e_2(P_4) = 2$, $e_3(P_4) = 3$ and $e_4(P_4) = 5$.

Point $P_5 = (0, 0, -\frac{1}{12}, \frac{1}{32})$ gives $\Omega_R(P_5) = -\frac{15}{6}$ which means that the exact solution at the point describes a Kantowski–Sachs universe. The eigenvalues of the linearized system are $e_1(P_5) = -\frac{15}{8}$, $e_2(P_5) = -\frac{3}{4}$, $e_3(P_5) = -\frac{5}{8}$ and $e_4(P_5) = \frac{3}{4}$, which means that P_5 is a saddle point.

Points $P_6^\pm = (0, x, \pm\sqrt{1-\lambda x^2}, \frac{1}{9}(1-\lambda x^2 \pm \sqrt{1-\lambda x^2}))$ are surfaces in the phase space where $\Omega_R(P_6) = 0$, which means that the points describe Bianchi I spacetimes. The points are real when $1 - x^2 \geq 0$. In the limit where $x^2 = \frac{1}{\lambda}$

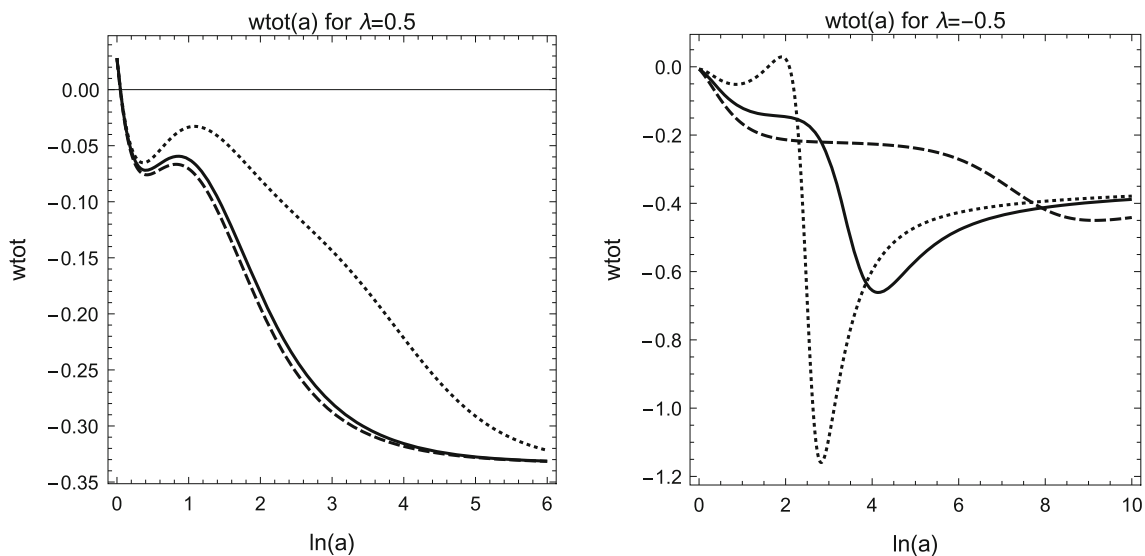


Fig. 2 Qualitative evolution of the parameter, w , for the equation of state of the effective fluid, for various initial conditions. The left-hand figure is for a positive value of $\lambda = 0.5$, while right-hand figure is for a negative value of $\lambda = -0.5$. We observe that the future attractor is the Milne universe, $w_{tot}(P_1) = -\frac{1}{3}$. The left-hand figure is for initial

conditions, $\Omega_{m0} = 0.75$, $\beta_0 = 0.02$, $x_0 = 0.4$, $\alpha_0 = 0$ (solid line), $\alpha_0 = 0.01$ (dashed line) and $\alpha_0 = -0.01$ (dotted line). The right-hand figure is for initial conditions, $\Omega_{m0} = 0.75$, $\beta_0 = 0.08$, $x_0 = 0.02$, $\alpha_0 = 0$ (solid line), $\alpha_0 = 0.01$ (dashed line) and $\alpha_0 = -0.01$ (dotted line)

Table 1 Stationary points and their stability for the Szekeres system in Weyl Integrable geometry with a dust fluid source

Point	$(\Omega_m, \mathbf{x}, \beta, \alpha)$	Ω_R	Spacetime	Stability
P_1	$(0, 0, 0, 0)$	-1	FLRW (Milne Universe)	Stable
P_2	$(0, 0, \frac{1}{6}, 0)$	$-\frac{3}{4}$	Kantowski–Sachs	Unstable
P_3	$(0, 0, -\frac{1}{3}, 0)$	0	Bianchi I	Unstable
P_4	$(0, 0, \frac{1}{3}, \frac{2}{9})$	0	Bianchi I	Unstable
P_5	$(0, 0, -\frac{1}{12}, \frac{1}{32})$	$-\frac{15}{6}$	Kantowski–Sachs	Unstable
P_6	$(0, x, \pm\sqrt{1-\lambda x^2}, \frac{1}{9}(1-\lambda x^2 \pm \sqrt{1-\lambda x^2}))$	0	Bianchi I	Unstable
P_7	$(1 - \frac{1}{6\lambda}, -\frac{1}{\sqrt{6\lambda}}, 0, 0)$	0	FLRW (spatially flat)	Unstable
P_8	$(-\frac{8}{3}\lambda, \sqrt{\frac{2}{3}}, 0, 0)$	$-1 - 2\lambda$	FLRW (open)	Unstable
P_9	$(-\frac{3\lambda(2\lambda+5)}{2(\lambda+2)^2}, \frac{1}{2+\lambda}\sqrt{\frac{3}{2}}, -\frac{1}{3} + \frac{1}{4+2\lambda}, \frac{3+4\lambda(\lambda+2)}{24(\lambda+2)^2})$	$-\frac{3(2\lambda+1)(2\lambda+5)}{4(\lambda+2)^2}$	Kantowski–Sachs	Unstable

the solution reduces to that of isotropic FLRW spacetime with a stiff fluid source. The eigenvalues of the linearized system are $e_1(P_6) = 0$, $e_2(P_6) = \frac{6+\sqrt{6x}}{2}$,

$$e_3(P_6^\pm) = 4 - \frac{4}{3}\lambda x^2 \mp \frac{\sqrt{1-\lambda x^2}}{2} + \frac{1}{6}\sqrt{81 + \lambda x^2 (64\lambda x^2 - 81 - 48\sqrt{1-\lambda x^2})},$$

$$e_4(P_6^\pm) = 4 - \frac{4}{3}\lambda x^2 \mp \frac{\sqrt{1-\lambda x^2}}{2} - \frac{1}{6}\sqrt{81 + \lambda x^2 (64\lambda x^2 - 81 - 48\sqrt{1-\lambda x^2})}.$$

Point $P_7 = (1 - \frac{1}{6\lambda}, -\frac{1}{\sqrt{6\lambda}}, 0, 0)$, describes a FLRW spacetime, $\Omega_R(P_7) = 0$, where the equation of state parameter for the effective fluid is $w_{tot}(P_7) = \frac{1}{6\lambda}$. This point is physically acceptable when $\lambda > \frac{1}{6}$, which means that $0 < w_{tot}(P_7) < 1$. The eigenvalues of the linearized system are $e_1(P_7) = 1 + \frac{1}{2\lambda}$, $e_2(P_7) = -\frac{3}{2} + \frac{1}{4\lambda}$, $e_3(P_7) = \frac{1}{4} + \frac{5}{72\lambda} + \frac{\sqrt{2916\lambda^2 + 1044\lambda - 191}}{72\lambda}$, $e_4(P_7) = \frac{1}{4} + \frac{5}{72\lambda} - \frac{\sqrt{2916\lambda^2 + 1044\lambda - 191}}{72\lambda}$. Whence it follows that the exact solution at point P_7 is unstable.

Point $P_8 = (-\frac{8}{3}\lambda, \sqrt{\frac{2}{3}}, 0, 0)$ is physical acceptable for $-\frac{3}{8} < \lambda < 0$. It describes a FLRW spacetime

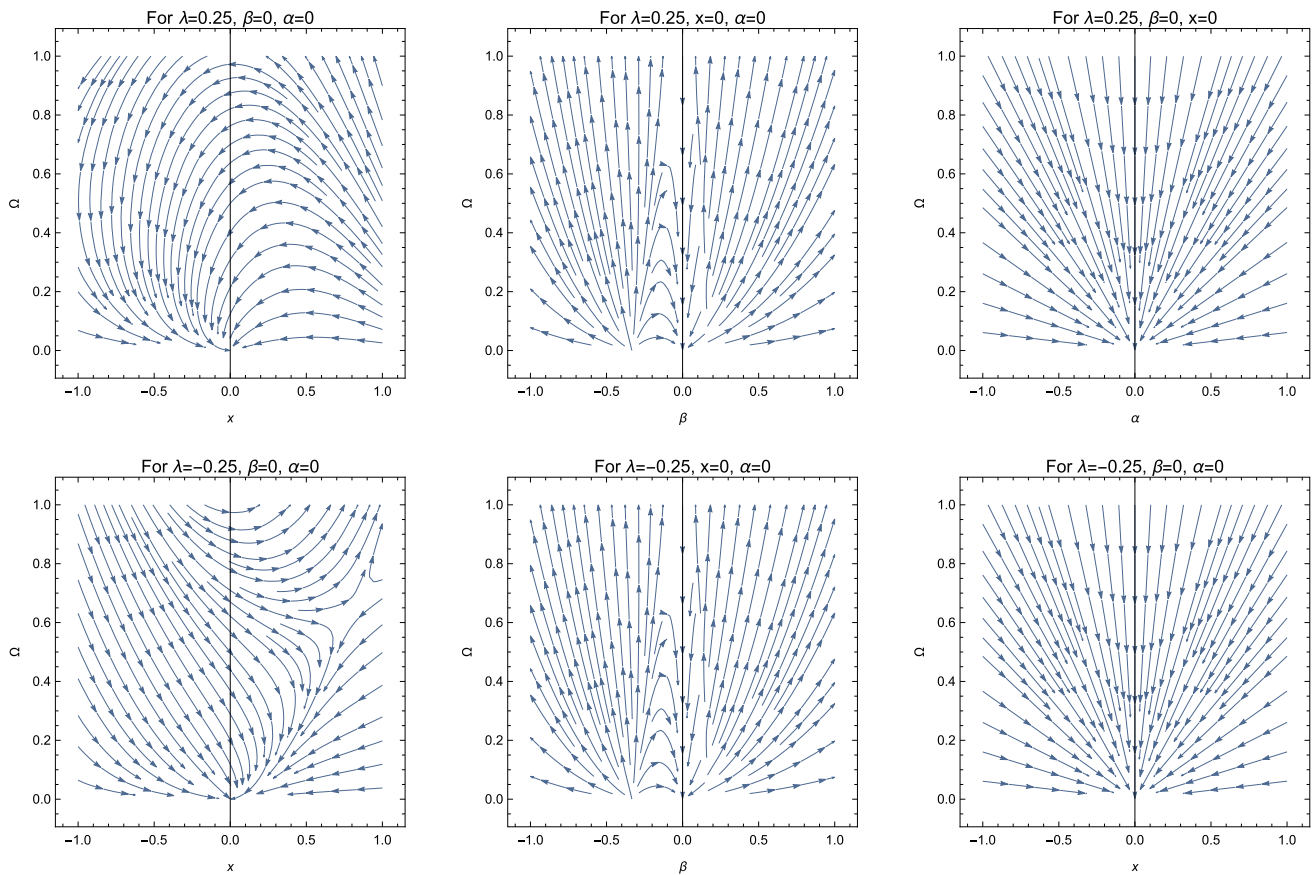


Fig. 3 Two-dimensional phase space portraits in the planes $\{x - \Omega\}$, $\{\beta - \Omega\}$ and $\{\alpha - \Omega\}$. The figures of the first row are for $\lambda = 0.25$, while the figures of the second row are for $\lambda = -0.25$. The unique attractor of the system is the Milne Universe

with spatial curvature $\Omega_R(P_8) = -1 - 2\lambda$, which is always negative for the accepted values of λ . The eigenvalues of the linearized system are derived to be, $e_1(P_8) = -1 + i\sqrt{2\lambda}$, $e_2(P_8) = -1 - i\sqrt{2\lambda}$, $e_3(P_8) = -\frac{1}{3} - \frac{8}{9}\lambda + \frac{1}{9}(\sqrt{64\lambda^2 + 156\lambda + 63})$, $e_4(P_8) = -\frac{1}{3} - \frac{8}{9}\lambda - \frac{1}{9}(\sqrt{64\lambda^2 + 156\lambda + 63})$ from which we conclude that the exact solution at P_8 is always unstable.

Point $P_9 = \left(-\frac{3\lambda(2\lambda+5)}{2(\lambda+2)^2}, \frac{1}{2+\lambda}\sqrt{\frac{3}{2}}, -\frac{1}{3} + \frac{1}{4+2\lambda}, \frac{3+4\lambda(\lambda+2)}{24(\lambda+2)^2}\right)$ describes a Kantowski–Sachs universe where $\Omega_R(P_9) = -\frac{3(2\lambda+1)(2\lambda+5)}{4(\lambda+2)^2}$. The point is physical acceptable for $-\frac{23-\sqrt{273}}{16} \leq \lambda < 0$ and $-\frac{5}{2} \leq \lambda < -\frac{23+\sqrt{273}}{16}$. The eigenvalues are calculated numerically, from which we infer that point P_9 is a saddle point.

The above results are summarized in Table 1. In Fig. 2, the qualitative behaviour of the equation of state parameter w_{tot} is presented. Moreover, two-dimensional phase portraits for the dynamical system (39a)–(39b) are presented in Figs. 3 and 4 where P_1 is the unique attractor. The plots are for positive and negative values of the coupling parameter λ .

4.3 Past attractors

When analyzing the dynamics of the system (39a)–(39d) towards the past, it is convenient to make a time reversal $\tau \mapsto -\tau$. In this case, we have the same points as before, but there is an overall change of sign in the eigenvalues. Then, the possible late-time attractors of the new system, given by

$$\Omega'_m = -\frac{1}{2}\Omega_m \left(\sqrt{6}x + 8\lambda x^2 + 72\beta^2 + 2(\Omega_m - 1) \right), \quad (41a)$$

$$x' = -\frac{1}{12\lambda} \left(2\lambda x (36\beta^2 + 4\lambda x^2 + \Omega_m - 4) - \sqrt{6}\Omega_m \right), \quad (41b)$$

$$\beta' = -\frac{1}{2} \left(6\beta^2(1 + 6\beta) + \beta\Omega - 2(\beta + 3\alpha) + 4\lambda\beta x^2 \right), \quad (41c)$$

$$\alpha' = -\frac{1}{2} \left(2\alpha \left(\Omega + 4\lambda x^2 - 1 + 9\beta(4\beta - 1) \right) - \beta \left(2\lambda x^2 + \Omega_m \right) \right), \quad (41d)$$

correspond to the past attractors of the original one. We study the points P_3 with coordinates $(\Omega_m, x, \beta, \alpha) =$

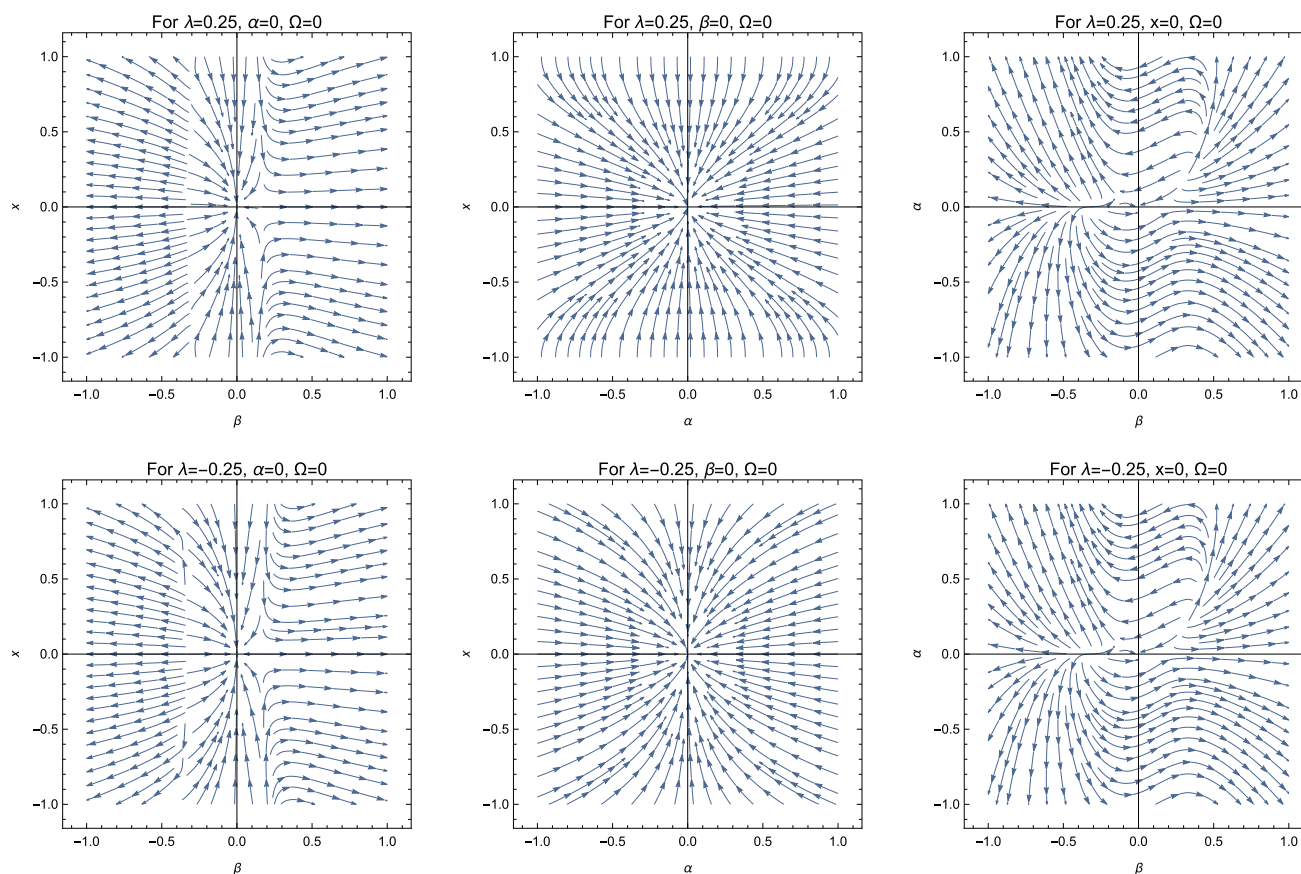


Fig. 4 Two-dimensional phase space portraits in the planes $\{\beta - x\}$, $\{\alpha - x\}$ and $\{\beta - \alpha\}$. The figures in the first row are for $\lambda = 0.25$, while the figures in the second row are for $\lambda = -0.25$. The unique attractor of the system is the Milne Universe

$(0, 0, -\frac{1}{3}, 0)$, $\Omega_R = 0$, and P_4 with coordinates $(\Omega_m, x, \beta, \alpha) = (0, 0, \frac{1}{3}, \frac{2}{9})$, $\Omega_R = 0$, corresponding to Bianchi I models, and we show they are unstable for the original system using the center manifold theorem (CMT). The detailed analysis of the CMT for these two points is presented in Appendix A.

5 Conclusions

In this work we found exact inhomogeneous spacetimes which generalize the Szekeres universes into the Weyl integrable theory. Specifically, we assume that the scalar field which defines the Weyl affine connection to be homogeneous such that the limit of FLRW exists. In such scenario, the only inhomogeneous spacetimes are those which belong to the FLRW (-like) solutions included in the family of LTB spacetimes. On the other hand, the Kantowski–Sachs family of solutions is homogeneous and anisotropic. For the inhomogeneous spacetimes, we were able to write in terms of quadratics the generic solution of the field equations.

In order to understand the dynamics and the evolution of the gravitational model we performed a detailed study of the past and future attractors. In particular, we defined

Hubble-normalized dimensionless variables. The field equations admit three stationary points which describe a spatially flat, an open, and a closed FLRW space where only the closed FLRW spacetime can be a future attractor, which gives the Milne universe. The other two isotropic solutions correspond to saddle points.

In addition, three homogeneous Kantowski–Sachs spacetimes are supported by the field equations which correspond to saddle points. There are three points which describe Bianchi I spacetimes; the exact solution at one of these points describes a Bianchi I spacetime with a stiff fluid, while the other two points describe vacuum Kasner solutions. The points which describe the Kasner solutions are sources while the third point is a saddle point.

For the sources we performed a detailed study on the past-system in order to investigate if the points are attractors for the past-system. Indeed with the application of the center manifold theorem we were able to prove that the Kasner solutions are past attractors of the field equations for $\lambda > 0$.

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Appendix A: CMT for points P_3 and P_4

Analysis of P_3

Introducing the coordinate transformations,

$$\alpha \mapsto v_3 - v_1, \beta \mapsto v_1 + \sqrt{\frac{2}{3}}\lambda v_2 - \frac{1}{3},$$

$$\Omega_m \mapsto -6\sqrt{6}\lambda v_2, x \mapsto u + v_2, \quad (\text{A1})$$

the equilibrium point P_3 is translated to the origin and the linearization matrix is transformed to its canonical real Jordan form.

Therefore, we obtain the equivalent dynamical system to (41), defined by

$$\begin{aligned} u' &= -\frac{1}{3}\lambda(u - 5v_2) \\ &\quad \times \left(2u^2 + 4uv_2 + v_2 \left(\sqrt{6}(12v_1 - 7) + 2v_2 \right) \right) \\ &\quad - 6v_1^2(u - 5v_2) + 4v_1(u - 5v_2) \\ &\quad - 4\lambda^2 v_2^2(u - 5v_2) + \sqrt{\frac{3}{2}}v_2(u + v_2), \\ v_1' &= \frac{1}{3} \left\{ -3v_1 \left(\lambda \left(2 \left((u + v_2)^2 - 6\lambda v_2^2 \right) - 5\sqrt{6}v_2 \right) + 6 \right) \right. \\ &\quad + 2\lambda^2 v_2 \left(\sqrt{6}(u + v_2)^2 - 18v_2 \right) \\ &\quad + \lambda \left((u + v_2)(2u + 5v_2) - 3\sqrt{6}v_2 \right) \\ &\quad \left. - 54v_1^3 + 9v_1^2 \left(5 - 2\sqrt{6}\lambda v_2 \right) + 12\sqrt{6}\lambda^3 v_2^3 + 9v_3 \right\}, \\ v_2' &= -\frac{1}{2}v_2 \left(8\lambda u^2 + u \left(16\lambda v_2 + \sqrt{6} \right) + 72v_1^2 \right. \\ &\quad \left. + 48v_1 \left(\sqrt{6}\lambda v_2 - 1 \right) \right. \\ &\quad \left. + 8\lambda(6\lambda + 1)v_2^2 + \sqrt{6}(1 - 28\lambda)v_2 + 6 \right), \\ v_3' &= 3v_1 \left(\lambda \left((u + v_2)^2 + 12\lambda v_2^2 - 5\sqrt{6}v_2 \right) + v_3 \left(11 - 8\sqrt{6}\lambda v_2 \right) \right) \end{aligned}$$

$$\begin{aligned} &+ \lambda^2 v_2 \left(\sqrt{6} \left((u + v_2)^2 + 4\lambda v_2^2 \right) - 18v_2 \right) \\ &- v_3 \left(\lambda \left(4 \left((u + v_2)^2 + 6\lambda v_2^2 \right) - 17\sqrt{6}v_2 \right) + 3 \right) \\ &+ \frac{1}{3}\lambda(u + v_2)(u + 4v_2) + 18v_1^3 - 18v_1^2 \left(-\sqrt{6}\lambda v_2 + 2v_3 + 1 \right). \end{aligned}$$

The eigen system of the origin is

$$\begin{pmatrix} -6 & -3 & -3 & 0 \\ \{0, 1, 0, 0\} & \{0, 1, 0, 1\} & \left\{0, -\sqrt{\frac{2}{3}}\lambda, 1, 0\right\} & \{1, 0, 0, 0\} \end{pmatrix}.$$

That is, the center manifold of the origin is tangent to the u -axis, and it is given locally by a graph

$$\begin{aligned} &\left\{ (u, v_1, v_2, v_3) \in \mathbb{R}^4 : v_i = h_i(u), h_i(0) = h_i'(0) = 0, \right. \\ &\quad \left. i = 1 \dots 4, |u| < \delta \right\}, \end{aligned} \quad (\text{A2})$$

which satisfies the differential equations

$$F(u, h_1, h_2, h_3)h_1'(u) + G_1(u, h_1, h_2, h_3) = 0, \quad (\text{A3})$$

$$2F(u, h_1, h_2, h_3)h_2'(u) + G_2(u, h_1, h_2, h_3) = 0, \quad (\text{A4})$$

$$2F(u, h_1, h_2, h_3)h_3'(u) + G_3(u, h_1, h_2, h_3) = 0, \quad (\text{A5})$$

where

$$\begin{aligned} F(u, h_1, h_2, h_3) &= 12h_1(u - 5h_2) \left(\sqrt{6}\lambda h_2 - 1 \right) \\ &\quad + 18h_1^2(u - 5h_2) \\ &\quad + \lambda(u - 5h_2) \left(h_2 \left(2h_2 + 4u - 7\sqrt{6} \right) + 2u^2 \right) \\ &\quad + 12\lambda^2(u - 5h_2)h_2^2 - 3\sqrt{\frac{3}{2}}h_2(h_2 + u), \\ G_1(u, h_1, h_2, h_3) &= -3h_1 \left(\lambda h_2 \left((2 - 12\lambda)h_2 + 4u - 5\sqrt{6} \right) \right. \\ &\quad \left. + 2\lambda u^2 + 6 \right) \\ &\quad + 9h_1^2 \left(5 - 2\sqrt{6}\lambda h_2 \right) \\ &\quad - 54h_1^3 + \lambda h_2 \left(h_2 \left(2\sqrt{6}(6\lambda + 1)\lambda h_2 \right. \right. \\ &\quad \left. \left. + 4\lambda \left(\sqrt{6}u - 9 \right) + 5 \right) \right. \\ &\quad \left. + u \left(2\sqrt{6}\lambda u + 7 \right) - 3\sqrt{6} \right) + 9h_3 + 2\lambda u^2, \\ G_2(u, h_1, h_2, h_3) &= -3h_2 \left(48h_1 \left(\sqrt{6}\lambda h_2 - 1 \right) + 72h_1^2 \right. \\ &\quad \left. + h_2 \left(8(6\lambda + 1)\lambda h_2 - 28\sqrt{6}\lambda + 16\lambda u + \sqrt{6} \right) \right. \\ &\quad \left. + u \left(8\lambda u + \sqrt{6} \right) + 6 \right), \\ G_3(u, h_1, h_2, h_3) &= h_1 \left(h_2 \left(\lambda \left(36u - 90\sqrt{6} \right) - 144\sqrt{6}\lambda h_3 \right) \right. \\ &\quad \left. + 18\lambda(12\lambda + 1)h_2^2 + 198h_3 + 18\lambda u^2 \right) \\ &\quad + h_1^2 \left(108\sqrt{6}\lambda h_2 - 216h_3 - 108 \right) + 108h_1^3 \\ &\quad + h_2 \left(6\lambda \left(17\sqrt{6} - 8u \right) h_3 \right. \\ &\quad \left. + 2\lambda u \left(3\sqrt{6}\lambda u + 5 \right) \right) \\ &\quad + h_2^2 \left(4\lambda \left(3\lambda \left(\sqrt{6}u - 9 \right) + 2 \right) \right. \\ &\quad \left. - 24\lambda(6\lambda + 1)h_3 \right) \\ &\quad + 6\sqrt{6}\lambda^2(4\lambda + 1)h_2^3 - 6h_3 \left(4\lambda u^2 + 3 \right) \\ &\quad + 2\lambda u^2. \end{aligned}$$

Using Taylor expansions, we propose as Ansätze,

$$\begin{pmatrix} h_1(u) \\ h_2(u) \\ h_3(u) \end{pmatrix} = \begin{pmatrix} a_1u^2 + a_2u^3 + a_3u^4 + a_4u^5 + a_5u^6 + a_6u^7 + a_7u^8 + a_8u^9 + a_9u^{10} + a_{10}u^{11} + a_{11}u^{12} + a_{12}u^{13} + \dots \\ b_1u^2 + b_2u^3 + b_3u^4 + b_4u^5 + b_5u^6 + b_6u^7 + b_7u^8 + b_8u^9 + b_9u^{10} + b_{10}u^{11} + b_{11}u^{12} + b_{12}u^{13} + \dots \\ c_1u^2 + c_2u^3 + c_3u^4 + c_4u^5 + c_5u^6 + c_6u^7 + c_7u^8 + c_8u^9 + c_9u^{10} + c_{10}u^{11} + c_{11}u^{12} + c_{12}u^{13} + \dots \end{pmatrix}$$

Substituting in (A3)–(A5), and equating the coefficients of equal powers of u , we obtain

$$\begin{aligned} a_1 &= \frac{\lambda}{6}, a_2 = 0, \\ a_3 &= \frac{\lambda^2}{24}, a_4 = 0, \\ a_5 &= \frac{\lambda^3}{48}, a_6 = 0, \\ a_7 &= \frac{5\lambda^4}{384}, a_8 = 0, a_9 = \frac{7\lambda^5}{768}, \\ a_{10} &= 0, a_{11} = \frac{7\lambda^6}{1024}, a_{12} = 0 \\ b_1 &= 0, b_2 = 0, b_3 = 0, \\ b_4 &= 0, b_5 = 0, b_6 = 0, b_7 = 0, \\ b_8 &= 0, b_9 = 0, b_{10} = 0, b_{11} = 0, b_{12} = 0, \\ c_1 &= \frac{\lambda}{9}, c_2 = 0, c_3 = \frac{\lambda^2}{18}, \\ c_4 &= 0, c_5 = \frac{\lambda^3}{36}, c_6 = 0, \\ c_7 &= \frac{5\lambda^4}{288}, c_8 = 0, \\ c_9 &= \frac{7\lambda^5}{576}, c_{10} = 0, c_{11} = \frac{7\lambda^6}{768}, c_{12} = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \alpha &\mapsto -\frac{\lambda u^2}{18} + \frac{\lambda^2 u^4}{72} + \frac{\lambda^3 u^6}{144} + \frac{5\lambda^4 u^8}{1152} + \frac{7\lambda^5 u^{10}}{2304} \\ &\quad + \frac{7\lambda^6 u^{12}}{3072} + \dots, \\ \beta &\mapsto -\frac{1}{3} + \frac{\lambda u^2}{6} + \frac{\lambda^2 u^4}{24} + \frac{\lambda^3 u^6}{48} + \frac{5\lambda^4 u^8}{384} \\ &\quad + \frac{7\lambda^5 u^{10}}{768} + \frac{7\lambda^6 u^{12}}{1024} + \dots, \\ \Omega_m &\mapsto 0, \\ x &\mapsto u, \end{aligned}$$

and we have the parametrization,

$$\begin{aligned} \dot{\phi} &= \sqrt{\frac{2}{3}}\theta(u+v_2) \sim \sqrt{\frac{2}{3}}\theta u + \mathcal{O}(u)^{14}, \\ \rho_m &= -2\sqrt{6}\theta^2\lambda v_2 e^{\frac{\phi}{2}} \sim \mathcal{O}(u)^{14}, \\ \sigma &= \frac{1}{3}\theta(3v_1 + \sqrt{6}\lambda v_2 - 1) \\ &\sim \theta \left(\frac{7\lambda^6 u^{12}}{1024} + \frac{7\lambda^5 u^{10}}{768} + \frac{5\lambda^4 u^8}{384} + \frac{\lambda^3 u^6}{48} \right. \\ &\quad \left. + \frac{\lambda^2 u^4}{24} + \frac{\lambda u^2}{6} - \frac{1}{3} \right) \\ &\quad + \mathcal{O}(u)^{14}, \\ E &= \theta^2(v_3 - v_1) \\ &\sim \theta^2\lambda \left(\frac{7\lambda^5 u^{12}}{3072} + \frac{7\lambda^4 u^{10}}{2304} + \frac{5\lambda^3 u^8}{1152} \right. \\ &\quad \left. + \frac{\lambda^2 u^6}{144} + \frac{\lambda u^4}{72} - \frac{u^2}{18} \right) \\ &\quad + \mathcal{O}(u)^{14}, \end{aligned}$$

where we choose $\lambda v_2 \geq 0$.

The dynamics on the center manifold of the origin are dictated by a gradient-like equation $u' = -\nabla U(u)$. For $\lambda > 0$, $\omega = u\sqrt{\lambda}$, the equation transforms to

$$\omega' = -\frac{\omega^{15}((7(63\omega^6 + 168\omega^4 + 352\omega^2 + 704)\omega^2 + 10560)\omega^2 + 33792)}{1572864}, \quad (\text{A6})$$

for which the origin is a degenerated minimum.

For $\lambda < 0$, $\omega = u\sqrt{-\lambda}$, the equation transforms to

$$\omega' = \frac{\omega^{15}(33792 - \omega^2(10560 - 7\omega^2(704 - \omega^2(352 - 21\omega^2(8 - 3\omega^2))))))}{1572864}, \quad (\text{A7})$$

for which the origin is a degenerated maximum.

Therefore, for $\lambda > 0$ (respectively, $\lambda < 0$) the center manifold, and hence, the origin of the dynamical system is a local attractor (respectively, a saddle). the origin of the dynamical system is a local attractor of system is P_3 , and for $\lambda < 0$ it is a saddle point.

Now, we take the time reversal back and work in terms of t . We deduce:

$$\begin{aligned}\dot{\theta} &= \theta^2 \left(-\frac{147\lambda^{12}u^{24}}{524288} - \frac{49\lambda^{11}u^{22}}{65536} \right. \\ &\quad \left. - \frac{77\lambda^{10}u^{20}}{49152} - \frac{77\lambda^9u^{18}}{24576} - \frac{55\lambda^8u^{16}}{8192} - \frac{11\lambda^7u^{14}}{512} - 1 \right) \\ &\sim -\theta^2 - \frac{11}{512} (\theta^2\lambda^7) u^{14} + \mathcal{O}(u^{16}), \\ \dot{u} &= \frac{1}{3}\theta \left(\frac{147\lambda^{12}u^{25}}{524288} + \frac{49\lambda^{11}u^{23}}{65536} \right. \\ &\quad \left. + \frac{77\lambda^{10}u^{21}}{49152} + \frac{77\lambda^9u^{19}}{24576} + \frac{55\lambda^8u^{17}}{8192} + \frac{11\lambda^7u^{15}}{512} \right) \\ &\sim \frac{11(\theta\lambda^7)u^{15}}{1536} + \mathcal{O}(u^{16}).\end{aligned}$$

The solutions can be expressed as:

$$\begin{aligned}\theta(t) &= \frac{1}{t-t_0} + \varepsilon c_1(t), \\ u(t) &= \frac{\varepsilon c_2(t)}{\sqrt[14]{-\ln(t-t_0)}}, \quad \varepsilon \ll 1,\end{aligned}\quad (A8)$$

where

$$\begin{aligned}c_1'(t) &= -\frac{2c_1(t)}{t-t_0} - \varepsilon c_1(t)^2 + \mathcal{O}(\varepsilon^{13}), \\ c_2'(t) &= \frac{c_2(t)}{14(t-t_0)\ln(t-t_0)} + \mathcal{O}(\varepsilon^{13}).\end{aligned}$$

Then,

$$\begin{aligned}c_1(t) &= \frac{1}{(t-t_0)(c_3(t-t_0)-\varepsilon)}, \\ c_2(t) &= c_4 \sqrt[14]{-\ln(t-t_0)}.\end{aligned}\quad (A9)$$

Finally,

$$\begin{aligned}\theta(t) &= -\frac{c_3}{c_3(t_0-t)+\varepsilon} \sim \frac{1}{t-t_0} + \frac{\varepsilon}{c_3(t-t_0)^2} \\ &\quad + \frac{\varepsilon^2}{(t-t_0)^3 c_3^2} + \mathcal{O}(\varepsilon^3), \quad u = c_4 \varepsilon,\end{aligned}\quad (A10a)$$

and

$$\begin{aligned}\sigma &= -\frac{1}{3(t-t_0)} - \frac{\varepsilon}{3((t-t_0)^2 c_3)} \\ &\quad + \frac{\left((t-t_0)^2 \lambda c_4^2 - \frac{2}{c_3^2} \right) \varepsilon^2}{6(t-t_0)^3} + \mathcal{O}(\varepsilon^3),\end{aligned}\quad (A11)$$

$$E = -\frac{(\lambda c_4^2) \varepsilon^2}{18(t-t_0)^2} + \mathcal{O}(\varepsilon^3), \quad (A12)$$

$$\dot{\phi} = \frac{\sqrt{\frac{2}{3}} c_4 \varepsilon}{t-t_0} + \frac{\sqrt{\frac{2}{3}} c_4 \varepsilon^2}{(t-t_0)^2 c_3} + \mathcal{O}(\varepsilon^3), \quad (A13)$$

$$\phi = \sqrt{\frac{2}{3}} c_4 \varepsilon \ln(t-t_0) - \frac{\sqrt{\frac{2}{3}} c_4 \varepsilon^2}{c_3(t-t_0)} + \mathcal{O}(\varepsilon^3), \quad (A14)$$

$$\rho_m = \frac{2\varepsilon^{14} (\sqrt{6} c_4^{14} K_0)}{(t-t_0)^2} + \mathcal{O}(\varepsilon^{15}), \quad (A15)$$

where c_3 and c_4 are integration constants, and we set $\lambda v_2 = -K_0 \varepsilon^{14}$, for a positive constant K_0 . For $\lambda > 0$, $\theta(t) \rightarrow \frac{1}{t-t_0}$ as $t \rightarrow 0$ ($\tau \rightarrow -\infty$). Hence, P_3 is associated with (an anisotropic) initial singularity.

5.1 Analysis of P_4

Introducing the coordinate transformation

$$\begin{aligned}\alpha &\mapsto \frac{3v_1}{2} + \sqrt{6}\lambda v_2 + \frac{3v_3}{5} + \frac{2}{9}, \\ \beta &\mapsto v_1 + \sqrt{6}\lambda v_2 + v_3 + \frac{1}{3}, \\ \Omega &\mapsto -6\sqrt{6}\lambda v_2, \quad x \mapsto u + v_2,\end{aligned}\quad (A16)$$

the equilibrium point P_4 is translated to the origin and the linearization matrix is transformed to its canonical real Jordan form.

Therefore, we obtain the equivalent dynamical system to (41), defined by

$$\begin{aligned}u' &= \frac{1}{6} \left\{ -2\lambda(u-5v_2) \right. \\ &\quad \times \left(2u^2 + 4uv_2 + v_2 \left(9\sqrt{6}(4v_1+4v_3+1) + 2v_2 \right) \right) \\ &\quad + 3 \left(v_2 \left(20(v_1+v_3)(3v_1+3v_3+2) + \sqrt{6}v_2 \right) \right. \\ &\quad \left. \left. - u \left(12v_1^2 + 8v_1(3v_3+1) - \sqrt{6}v_2 + 4v_3(3v_3+2) \right) \right) \right. \\ &\quad \left. - 216\lambda^2 v_2^2 (u-5v_2) \right\},\end{aligned}$$

$$\begin{aligned}v_1' &= \frac{1}{405} \left\{ -15v_1 \left(3\lambda \left(38(u+v_2)^2 + 3888\lambda v_2^2 + 145\sqrt{6}v_2 \right) \right. \right. \\ &\quad \left. + v_3 \left(3888\sqrt{6}\lambda v_2 + 669 \right) + 1944v_3^2 + 55 \right) \\ &\quad + 6 \left[-15\lambda^2 v_2 \left(\sqrt{6} \left((u+v_2)^2 + 324\lambda v_2^2 \right) + 60v_2 \right) \right. \\ &\quad \left. + v_3 \left(-15\lambda \left((u+v_2)^2 + 972\lambda v_2^2 + 32\sqrt{6}v_2 \right) - 16 \right) \right. \\ &\quad \left. - 30v_3^2 \left(81\sqrt{6}\lambda v_2 + 11 \right) - 810v_3^3 \right] \\ &\quad \left. - 10\lambda(7u-47v_2)(u+v_2) - 19440v_1^3 \right. \\ &\quad \left. - 45v_1^2 \left(972\sqrt{6}\lambda v_2 + 972v_3 + 179 \right) \right\},\end{aligned}\quad (A17)$$

$$\begin{aligned}v_2' &= \frac{1}{2} v_2 \left\{ -u \left(8\lambda u + \sqrt{6} \right) \right. \\ &\quad \left. - v_2 \left(16\lambda u + \sqrt{6}(36\lambda(4v_1+4v_3+1) + 1) \right) \right. \\ &\quad \left. - 6(2v_1+2v_3+1)(6v_1+6v_3+1) - 8\lambda(54\lambda+1)v_2^2 \right\}, \\ v_3' &= \frac{1}{162} \left\{ 3v_1 \left(60\lambda \left(2(u+v_2)^2 + 486\lambda v_2^2 + \sqrt{6}v_2 \right) \right. \right.\end{aligned}$$

$$\begin{aligned}
& +6v_3(972\sqrt{6}\lambda v_2 - 155) + 972v_3^2 - 25) \\
& + 360\lambda^2 v_2(\sqrt{6}(u + v_2)^2 + 81\lambda v_2^2) + 6v_2) \\
& - 6v_3(3\lambda(16(u + v_2)^2 - 2916\lambda v_2^2 + 71\sqrt{6}v_2) + 134) \\
& - 10\lambda(8u - 19v_2)(u + v_2) + 4860v_1^3 \\
& + 36v_1^2(405\sqrt{6}\lambda v_2 + 243v_3 - 5) \\
& + 18v_3^2(162\sqrt{6}\lambda v_2 - 145) - 972v_3^3.
\end{aligned}$$

The eigen system of the origin is

$$\left(\begin{array}{cccc} -5 & -3 & -2 & 0 \\ \{0, \frac{2}{25}, 0, 1\} & \{0, 0, 1, 0\} & \{0, -\frac{32}{5}, 0, 1\} & \{1, 0, 0, 0\} \end{array} \right).$$

That is, the center manifold of the origin is tangent to the u -axis, and it is given locally by a graph

$$\begin{aligned}
& \left\{ (u, v_1, v_2, v_3) \in \mathbb{R}^4 : v_i = h_i(u), h_i(0) = h'_i(0) = 0, \right. \\
& \left. i = 1 \dots 4, |u| < \delta \right\}, \quad (\text{A18})
\end{aligned}$$

which satisfies the differential equations

$$F(u, h_1, h_2, h_3)h'_1(u) + G_1(u, h_1, h_2, h_3) = 0, \quad (\text{A19})$$

$$F(u, h_1, h_2, h_3)h'_2(u) + G_2(u, h_1, h_2, h_3) = 0, \quad (\text{A20})$$

$$F(u, h_1, h_2, h_3)h'_3(u) + G_3(u, h_1, h_2, h_3) = 0, \quad (\text{A21})$$

where

$$\begin{aligned}
F(u, h_1, h_2, h_3) &= h_1 \left(h_2 \left(4 \left(3\sqrt{6}\lambda u - 5 \right) - 60h_3 \right) \right. \\
&\quad \left. - 60\sqrt{6}\lambda h_2^2 + 12uh_3 + 4u \right) + h_1^2(6u - 30h_2) \\
&\quad + h_2 \left(4h_3 \left(3\sqrt{6}\lambda u - 5 \right) - 30h_3^2 \right. \\
&\quad \left. - 2\lambda u^2 + \sqrt{\frac{3}{2}}(6\lambda - 1)u \right) \\
&\quad + h_2^2 \left(-60\sqrt{6}\lambda h_3 - \sqrt{\frac{3}{2}}(30\lambda + 1) \right. \\
&\quad \left. + 6\lambda(6\lambda - 1)u - \frac{10}{3}\lambda(54\lambda + 1)h_2^3 + 6uh_2^2 \right. \\
&\quad \left. + 4uh_3 + \frac{2\lambda u^3}{3} \right), \\
G_1(u, h_1, h_2, h_3) &= h_1 \left(h_2 \left(-144\sqrt{6}\lambda h_3 - \frac{1}{9}\lambda(76u + 145\sqrt{6}) \right) \right. \\
&\quad \left. - \frac{2}{9}\lambda(1944\lambda + 19)h_2^2 - 72h_3^2 \right. \\
&\quad \left. - \frac{223h_3}{9} - \frac{38\lambda u^2}{9} - \frac{55}{27} \right) \\
&\quad + h_1^2 \left(-108\sqrt{6}\lambda h_2 - 108h_3 - \frac{179}{9} \right) - 48h_1^3
\end{aligned}$$

$$\begin{aligned}
& + h_2^2 \left(\frac{2}{81}\lambda(47 - 18\lambda(\sqrt{6}u + 30)) \right. \\
& \left. - \frac{2}{9}\lambda(972\lambda + 1)h_3 \right) \\
& + h_2 \left(-36\sqrt{6}\lambda h_3^2 - \frac{4}{9}\lambda(u + 16\sqrt{6})h_3 \right. \\
& \left. - \frac{2}{81}\lambda u(9\sqrt{6}\lambda u - 40) \right) \\
& - \frac{2}{3}\sqrt{\frac{2}{3}}\lambda^2(324\lambda + 1)h_2^3 \\
& - \frac{2}{135}h_3(15\lambda u^2 + 16) \\
& - 12h_3^3 - \frac{44h_3^2}{9} - \frac{14\lambda u^2}{81},
\end{aligned}$$

$$\begin{aligned}
G_2(u, h_1, h_2, h_3) &= h_2^2 \left(-72\sqrt{6}\lambda h_1 - 72\sqrt{6}\lambda h_3 \right. \\
&\quad \left. - \sqrt{\frac{3}{2}}(36\lambda + 1) - 8\lambda u \right) \\
&\quad + h_2(h_1(-72h_3 - 24) \\
&\quad - 36h_1^2 - 36h_3^2 - 24h_3 \\
&\quad + \frac{1}{2}(-u(8\lambda u + \sqrt{6}) - 6)) \\
&\quad - 4\lambda(54\lambda + 1)h_2^3, \\
G_3(u, h_1, h_2, h_3) &= h_1 \left(h_2 \left(108\sqrt{6}\lambda h_3 + \frac{10}{9}\lambda(4u + \sqrt{6}) \right) \right. \\
&\quad + \frac{20}{9}\lambda(243\lambda + 1)h_2^2 + 18h_3^2 - \frac{155h_3}{9} \\
&\quad + \frac{5}{54}(24\lambda u^2 - 5)) \\
&\quad + h_1^2 \left(90\sqrt{6}\lambda h_2 + 54h_3 - \frac{10}{9} \right) + 30h_1^3 \\
&\quad + h_2^2 \left(\frac{4}{9}\lambda(729\lambda - 4)h_3 \right. \\
&\quad + \frac{5}{81}\lambda(72\lambda(\sqrt{6}u + 3) + 19)) \\
&\quad + h_2 \left(18\sqrt{6}\lambda h_3^2 - \frac{1}{9}\lambda(32u + 71\sqrt{6})h_3 \right. \\
&\quad + \frac{5}{81}\lambda u(36\sqrt{6}\lambda u + 11)) \\
&\quad + \frac{20}{3}\sqrt{\frac{2}{3}}\lambda^2(81\lambda + 1)h_2^3 \\
&\quad - \frac{2}{27}h_3(24\lambda u^2 + 67) \\
&\quad - 6h_3^3 - \frac{145h_3^2}{9} - \frac{40\lambda u^2}{81}.
\end{aligned}$$

Using Taylor expansion we propose as Ansätze

$$\begin{pmatrix} h_1(u) \\ h_2(u) \\ h_3(u) \end{pmatrix} = \begin{pmatrix} a_1u^2 + a_2u^3 + a_3u^4 + a_4u^5 + a_5u^6 + a_6u^7 + a_7u^8 + a_8u^9 + a_9u^{10} + a_{10}u^{11} + a_{11}u^{12} + a_{12}u^{13} + \dots \\ b_1u^2 + b_2u^3 + b_3u^4 + b_4u^5 + b_5u^6 + b_6u^7 + b_7u^8 + b_8u^9 + b_9u^{10} + b_{10}u^{11} + b_{11}u^{12} + b_{12}u^{13} + \dots \\ c_1u^2 + c_2u^3 + c_3u^4 + c_4u^5 + c_5u^6 + c_6u^7 + c_7u^8 + c_8u^9 + c_9u^{10} + c_{10}u^{11} + c_{11}u^{12} + c_{12}u^{13} + \dots \end{pmatrix}. \quad (\text{A22})$$

Hence it follows

$$\begin{aligned} a_1 &= -\frac{2\lambda}{27}, a_2 = 0, a_3 = \frac{\lambda^2}{81}, a_4 = 0, a_5 = \frac{\lambda^3}{162}, a_6 = 0, \\ a_7 &= \frac{5\lambda^4}{1296}, a_8 \rightarrow 0, a_9 = \frac{7\lambda^5}{2592}, a_{10} = 0, \\ a_{11} &= \frac{7\lambda^6}{3456}, a_{12} = 0, \\ b_1 &= 0, b_2 = 0, b_3 = 0, b_4 = 0, b_5 = 0, b_6 = 0, \\ b_7 &= 0, b_8 = 0, b_9 = 0, b_{10} = 0, b_{11} = 0, b_{12} = 0, \\ c_1 &= -\frac{5\lambda}{54}, c_2 = 0, c_3 = -\frac{35\lambda^2}{648}, \\ c_4 &= 0, c_5 = -\frac{35\lambda^3}{1296}, c_6 = 0, \\ c_7 &= -\frac{175\lambda^4}{10368}, c_8 = 0, c_9 = -\frac{245\lambda^5}{20736}, \\ c_{10} &= 0, c_{11} = -\frac{245\lambda^6}{27648}, c_{12} = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \alpha &\mapsto \frac{2}{9} - \frac{\lambda u^2}{6} - \frac{\lambda^2 u^4}{72} - \frac{\lambda^3 u^6}{144} - \frac{5\lambda^4 u^8}{1152} \\ &\quad - \frac{7\lambda^5 u^{10}}{2304} - \frac{7\lambda^6 u^{12}}{3072} + \dots, \\ \beta &\mapsto \frac{1}{3} - \frac{\lambda u^2}{6} - \frac{\lambda^2 u^4}{24} - \frac{\lambda^3 u^6}{48} - \frac{5\lambda^4 u^8}{384} \\ &\quad - \frac{7\lambda^5 u^{10}}{768} - \frac{7\lambda^6 u^{12}}{1024} + \dots, \\ \Omega_m &\mapsto 0, \\ x &\mapsto u, \end{aligned}$$

and we have the parametrization

$$\begin{aligned} \dot{\phi} &= \sqrt{\frac{2}{3}}\theta u + \mathcal{O}(u)^{14}, \\ \rho_m &= -2\sqrt{6}\theta^2\lambda v_2 e^{\phi/2} \sim \mathcal{O}(u)^{14}, \\ \sigma &= \theta \left(-\frac{7\lambda^6 u^{12}}{1024} - \frac{7\lambda^5 u^{10}}{768} - \frac{5\lambda^4 u^8}{384} - \frac{\lambda^3 u^6}{48} \right. \\ &\quad \left. - \frac{\lambda^2 u^4}{24} - \frac{\lambda u^2}{6} + \frac{1}{3} \right) + \mathcal{O}(u)^{14}, \end{aligned}$$

$$E = \theta^2 \left(-\frac{7\lambda^6 u^{12}}{3072} - \frac{7\lambda^5 u^{10}}{2304} - \frac{5\lambda^4 u^8}{1152} - \frac{\lambda^3 u^6}{144} - \frac{\lambda^2 u^4}{72} - \frac{\lambda u^2}{6} + \frac{2}{9} \right) + \mathcal{O}(u)^{14},$$

where we choose $\lambda v_2 \geq 0$.

The dynamics on the center manifold of the origin are dictated by a gradient-like equation $u' = -\nabla U(u)$. For $\lambda > 0$, $\omega = u\sqrt{\lambda}$, the equation transforms to

$$\omega' = -\frac{\omega^{15}((7(63\omega^6 + 168\omega^4 + 352\omega^2 + 704)\omega^2 + 10560)\omega^2 + 33792)}{1572864}$$

for which the origin is a degenerated minimum. For $\lambda < 0$, $\omega = u\sqrt{-\lambda}$, the equation transforms to

$$\omega' = \frac{\omega^{15}((-7(63\omega^6 - 168\omega^4 + 352\omega^2 - 704)\omega^2 - 10560)\omega^2 + 33792)}{1572864},$$

for which the origin is a degenerated maximum. Therefore, for $\lambda > 0$ (respectively, $\lambda < 0$) the center manifold, and hence, the origin of the system, is a local attractor (respectively, a saddle). In the original variables mean that for $\lambda > 0$ the past attractor of the dynamical system is P_4 , and for $\lambda < 0$ is a saddle point. That is exactly the same dynamics as for P_3 . However, as we will see shortly, the physical solution, although it is Bianchi I, has a different asymptotic expansion.

Now, we take the time reversal back and work in terms of t . Hence

$$\begin{aligned} \dot{\theta} &= \theta^2 \left(-\frac{147\lambda^{12}u^{24}}{524288} - \frac{49\lambda^{11}u^{22}}{65536} - \frac{77\lambda^{10}u^{20}}{49152} - \frac{77\lambda^9u^{18}}{24576} \right. \\ &\quad \left. - \frac{55\lambda^8u^{16}}{8192} - \frac{11\lambda^7u^{14}}{512} - 1 \right) \\ &\sim -\theta^2 - \frac{11}{512}u^{14}(\theta^2\lambda^7) + \mathcal{O}(u^{16}), \\ \dot{u} &= \frac{\theta}{3} \left(\frac{147\lambda^{12}u^{25}}{524288} + \frac{49\lambda^{11}u^{23}}{65536} + \frac{77\lambda^{10}u^{21}}{49152} \right. \\ &\quad \left. + \frac{77\lambda^9u^{19}}{24576} + \frac{55\lambda^8u^{17}}{8192} + \frac{11\lambda^7u^{15}}{512} \right) \\ &\sim \frac{11(\theta\lambda^7)u^{15}}{1536} + \mathcal{O}(u^{16}). \end{aligned}$$

As before,

$$\theta(t) = -\frac{c_3}{c_3(t-t_0) + \varepsilon} \sim \frac{1}{t-t_0} + \frac{\varepsilon}{c_3(t-t_0)^2} + \frac{\varepsilon^2}{(t-t_0)^3 c_3^2} + \mathcal{O}(\varepsilon^3), \quad u = c_4 \varepsilon, \quad (\text{A23a})$$

however for this point

$$\sigma = \frac{1}{3(t-t_0)} + \frac{\varepsilon}{3(t-t_0)^2 c_3} + \frac{\left(\frac{2}{c_3^2} - (t-t_0)^2 \lambda c_4^2\right) \varepsilon^2}{6(t-t_0)^3} + \mathcal{O}(\varepsilon^3), \quad (\text{A24})$$

$$E = \frac{2}{9(t-t_0)^2} + \frac{4\varepsilon}{9(t-t_0)^3 c_3} + \frac{\left(\frac{4}{c_3^2} - (t-t_0)^2 \lambda c_4^2\right) \varepsilon^2}{6(t-t_0)^4} + \mathcal{O}(\varepsilon^3), \quad (\text{A25})$$

$$\dot{\phi} = \frac{\sqrt{\frac{2}{3}} c_4 \varepsilon}{t-t_0} + \frac{\sqrt{\frac{2}{3}} c_4 \varepsilon^2}{(t-t_0)^2 c_3} + \mathcal{O}(\varepsilon^3), \quad (\text{A26})$$

$$\phi = \sqrt{\frac{2}{3}} c_4 \varepsilon \ln(t-t_0) - \frac{\sqrt{\frac{2}{3}} c_4 \varepsilon^2}{c_3(t-t_0)} + \mathcal{O}(\varepsilon^3), \quad (\text{A27})$$

$$\rho_m = \frac{2\sqrt{6} c_4^{14} K_0 \varepsilon^{14}}{(t-t_0)^2} + \mathcal{O}(\varepsilon^{15}). \quad (\text{A28})$$

where c_3 and c_4 are integration constants, and we set $\lambda v_2 = -K_0 \varepsilon^{14}$, for a positive constant K_0 . As for P_3 , for $\lambda > 0$, $\theta(t) \rightarrow \frac{1}{t-t_0}$ as $t \rightarrow 0$ ($\tau \rightarrow -\infty$). Hence, P_4 it is associated with (an anisotropic) initial singularity. However, as per the physical solution referred to, this is a different solution with different asymptotic expansions for σ , E .

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